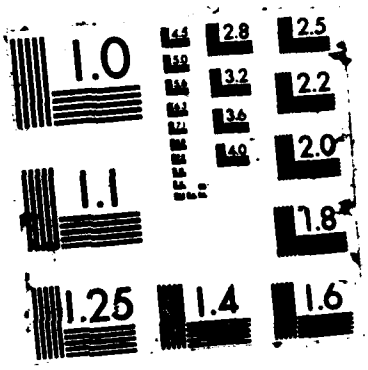
AD-A185 718 ASYMPTOTIC NORMALITY OF POLY-T DENSITIES WITH BAYESIAN APPLICATIONS(U) DEWITT WALLACE RESEARCH LAB NEW YORK G Y WONG 01 OCT 87 ONR-87-1 N00014-85-K-0485 1/1

UNCLASSIFIED F/G 12/3 NL


1.0
1.1
1.25
1.4
1.6
2.8
3.2
3.6
4.0
2.5
2.2
2.0
1.8

DTIC FILE COPY

AD-A185 718

(12)

Research Report ONR-87-1

ASYMPTOTIC NORMALITY OF POLY-T DENSITIES WITH BAYESIAN APPLICATIONS

George Y. Wong

Division of Biostatistics

Memorial Sloan-Kettering Cancer Center

New York, New York 10021

October, 1987

DTIC
ELECTE
OCT 2 3 1987
S D
CD

Prepared under contract NO. N00014-85-K-0485, NR 150-536
with the Personnel and Training Research Programs
Psychological Sciences Division
Office of Naval Research

Approved for public release; distribution unlimited.
Reproduction in whole or in part is permitted for
any purpose of the United States Government.

87 10 15 099

unclassified
SECURITY CLASSIFICATION OF THIS PAGE

ADA185718

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY unclassified	3. DISTRIBUTION / AVAILABILITY OF REPORT Approval for public release; distribution unlimited
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION George Y. Wong 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Office of Naval Research
6c. ADDRESS (City, State, and ZIP Code) Memorial Sloan-Kettering Cancer Center 1275 York Avenue New York, New York 10021	7b. ADDRESS (City, State, and ZIP Code) 800 North Quincy Street Arlington, VA 22217
8a. NAME OF FUNDING / SPONSORING ORGANIZATION Personnel & Training Research 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER N000 14-85-K-0485
8c. ADDRESS (City, State, and ZIP Code) Office of Naval Research, Code 442 800 North Quincy Street Arlington, VA 22217	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. NR150-536 / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)

Asymptotic Normality of Poly-t Densities With Bayesian Applications

12. PERSONAL AUTHOR(S) George Y. Wong

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
technical report	FROM 9/1/85 TO 8/1/87	1987 October 1	20

16. SUPPLEMENTARY NOTATION

(micron (underlined))

17. COSATI CODES		
FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

a certain

M (underlined)

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

A poly-t density is a density which is proportional to a product of at least two t-like factors, each of which is of the form $(1 + (x-\mu)^T M (x-\mu))^{-d/2}$ where d is a positive number, μ is an arbitrary location vector and M is a symmetric semi-positive definite scale matrix. In general, M is a function of d. Such a density arises, for example, in the Bayesian analysis of a linear model with a normal error term, independent normal priors on the linear parameters and inverted-gamma priors on the variance components. A theorem about the asymptotic normality of the density as a subset of the individual d's tend to infinity is proved under very general conditions. A corollary specifically related to the Bayesian linear model is also given. Detailed results are illustrated in the familiar Bayesian multiple linear regression model with two variance components. The Tiao-Zellner expansion for approximating the particular poly-t form involving two proper multivariate t factors is extended to the case of two arbitrary t-like factors.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL Dr. Charles Davis	22b. TELEPHONE (Include Area Code) 202/696-4046 — 22c. OFFICE SYMBOL

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete. — SECURITY CLASSIFICATION OF THIS PAGE

DTIC COPY INSPECTED 8

Accesion For	
NTIS CRA&I	☑
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution /	
Availability Codes	
Dist	Avail and / or Special
A-1	

ASYMPTOTIC NORMALITY OF POLY-T DENSITIES WITH BAYESIAN APPLICATIONS

GEORGE Y. WONG

DIVISION OF BIOSTATISTICS
MEMORIAL SLOAN-KETTERING CANCER CENTER
NEW YORK, NY 10021

Key Words and Phrases: poly-t density; asymptotic normality; generalized Tiao-Zellner expansion; Bayesian linear model; posterior inference of linear parameters.

ABSTRACT

A poly-t density is a density which is proportional to a product of at least two t-like factors, each of which is of the form $\{1 + (\underline{x} - \underline{\mu})^T \underline{M} (\underline{x}-\underline{\mu})\}^{-d/2}$ where d is a positive number, $\underline{\mu}$ is an arbitrary location vector and $\underline{M}$ is a symmetric semi-positive definite scale matrix. In general, $\underline{M}$ is a function of d. Such a density arises, for example, in the Bayesian analysis of a linear model with a normal error term, independent normal priors on the linear parameters and inverted-gamma priors on the variance components. A theorem about the asymptotic normality of the density as a subset of the individual d's tend to infinity is proved under very general conditions. A corollary specifically related to the Bayesian linear model is also given. Detailed results are illustrated in the familiar Bayesian multiple linear regression model with two variance components. The Tiao-Zellner expansion for approximating the particular poly-t form involving two proper multivariate t factors is extended to the case of two arbitrary t-like factors.

1. INTRODUCTION

A p-dimensional random vector $\underline{X}$ is said to have a poly-t distribution if its density is proportional to a product of $L \geq 2$ "t-like" factors, or

$$f(\underline{x}|d_1,\ldots,d_L) \propto \prod_{k=1}^{L} \{1+(\underline{x}-\underline{\mu}_k)^T \underline{M}_k(d_k)\ (\underline{x}-\underline{\mu}_k)\}^{-d_k/2} \qquad (1)$$

where $d_k>0$, $\underline{\mu}_k$ is a pxl location vector and $\underline{M}_k(d_k)$ is a pxp symmetric semi-positive definite scale matrix, k=1,...,L. To ensure that the right-hand side of (1), which we denote by $g(\underline{x}|d_1,\ldots,d_L)$, is normalizable, we must require that $d_1+\ldots+d_L>p$ and $\underline{M}_1(d_1)+\ldots+\underline{M}_L(d_L)$ be positive definite (Dickey 1986). The normalizing constant $\int g(\underline{x}|d_1,\ldots,d_L)d\underline{x}$, however, cannot be expressed in a simple closed form. When L is smaller than p, Dickey showed that the normalizing constant can be expressed in an (L-1)-fold integral.

A useful form of density (1) is obtained by letting

$$d_k=\nu_k + m_k,\ \underline{M}_k(d_k) = \underline{M}_k/\nu_k \qquad (2)$$

for some $\nu_k>0$, m_k such that $\nu_k+m_k>0$, and $\underline{M}_k$ symmetric semi-positive definite. When $m_k=p$ and $\underline{M}_k$ is positive definite, then the kth t-like factor is proportional to a proper multivariate t density with ν_k degrees of freedom. Such a poly-t density form plays an important role in the Bayesian analysis of a general linear model with a normal error term. The Bayesian approach considered here is described as follows: (1) A vague prior is imposed on a subset of the linear parameters; (2) The rest of the parameters are partitioned into different subsets, and an exchangeable normal prior with mean 0 and an unknown variance component is imposed on each subset independently; and (3) The prior variances and the error variance are given independent inverted gamma distributions. It can be shown that the posterior density of the linear parameters is of the above poly-t density form. Details about poly-t posterior

distributions can be found in Dickey (1974), Drèze (1977), Rajagopalan and Broemeling (1983), Broemeling and Abdullah (1984), and Broemeling (1985).

In this article, we are concerned with the asymptotic normality of a poly-t density as some or all of $d_1,\ldots,d_L$ become large. The asymptotic normal density can be used to approximate the poly-t density directly. Also, it can serve as an important sampling distribution for the Monte Carlo evaluation of any probability statement about $\underline{X}$, or any moment of $\underline{X}$. Despite its usefullness in Bayesian linear modeling, the asymptotic normality of a poly-t density has never been formally established. In Section 2, we prove the normality result for the poly-t density (1) under fairly general conditions. As will be seen there, the proof is much harder than that in the case of a single multivariate t density; the difficulty is mainly due to the fact that the normalizing constant of a poly-t density is in terms of an intergral expression.

In Section 3, we illustrate the asymptotic results on a Bayesian linear model with two variance components. There we show that the appropriate asymptotic normal density may be made the leading term in a generalization of the Tiao-Zellner expansion (1964) to approximate the posterior density of the linear parameters.

2. MAIN RESULTS

We show that under some fairly general conditions, the poly-t density (1) converges to a proper multivariate noraml density. It follows from a result of Scheffe (1947) that the poly-t variable actually converges in distribution to the multivariate normal variable with the limiting normal density. We also give a corollary on a special form of (1) which is particularly useful in Bayesian linear modeling. For convenience, we write $\underline{A}>0$ and $\underline{A}>0$ to mean that the symmetric matrix $\underline{A}$ is semi-positive definite and positive definite, respectively. Also, we denote the density of a multivariate normal density

with the mean $\underline{\mu}$ and covariance matrix $\underline{\Sigma}$ by $f_N(\cdot|\underline{\mu},\underline{\Sigma})$.

Theorem For the general poly-t density (1), assume that

(1) $d_k\underline{M}_k(d_k)$ converges to $\underline{M}_k \geq 0$ as $d_k \to \infty$, k=1,..., L, and $\underline{M} = \underline{M}_1 + \ldots + \underline{M}_L > 0$, and

(2) there exist semi-positive definite matrices $\underline{B}_1, \ldots, \underline{B}_L$ such that

$\underline{B}_1 + \ldots + \underline{B}_L > 0$ and $d_k\underline{M}_k(d_k) - \underline{B}_k \geq 0$ for all d_k sufficiently large.

Then as $d_1, \ldots, d_L$ all tend to infinity, $f(\underline{x}|d_1,..,d_L)$ will tend to the limiting density $f_N(\underline{x}|\underline{\mu},\underline{M}^{-1})$, where $\underline{\mu} = \underline{M}^{-1}(\underline{M}_1\underline{\mu}_1 + \ldots + \underline{M}_L\underline{\mu}_L)$.

Proof. Without any loss of generality, we prove the theorem for the case L=2. To show that $f(\underline{x}|d_1,d_2)$ converges to $f_N(\underline{x}|\underline{\mu}, \underline{M}^{-1})$ as $d_1, d_2 \to \infty$, it suffices to show that (i) $g(\underline{x}|d_1,d_2)$ tends to a limit $g(\underline{x})$ proportional to $f_N(\underline{x}|\underline{\mu},\underline{M}^{-1})$, and (ii) $\int g(\underline{x}|d_1,d_2) \to \int g(\underline{x})d\underline{x}$, as $d_1, d_2 \to \infty$.

To prove (i), we make use of assumption (1) and obtain

$$g(\underline{x}|d_1,d_2) = \prod_{k=1}^{2} \{1 + Q_k/d_k + o(1/d_k)\}^{-d_k/2},$$

where $Q_k = (\underline{x}-\underline{\mu}_k)^T\underline{M}_k(\underline{x}-\underline{\mu}_k)$, k = 1, 2. It follows that $g(\underline{x}|d_1,d_2)$ tends to

$$g(\underline{x}) = \exp\{-(\underline{\mu}_1-\underline{\mu}_2)^T\underline{M}_1\underline{M}^{-1}\underline{M}_2(\underline{\mu}_1-\underline{\mu}_2)/2\} \times \exp\{-(\underline{x}-\underline{\mu})^T\underline{M}(\underline{x}-\underline{\mu})/2\},$$

which is proportional to $f_N(\underline{x}|\underline{\mu},\underline{M}^{-1})$.

To prove (ii), we show that for d_1 and d_2 sufficiently large, $g(\underline{x}|d_1,d_2)$ is dominated by an integrable function $h(\underline{x}|d_1,d_2)$, which monotonically decreases to an integrable function $h(\underline{x})$; moreover, $\int h(\underline{x}|d_1,d_2)d\underline{x}$ also decreases monotonically to $\int h(\underline{x})d\underline{x}$. Then using a generalization of the Lebesque Dominated Convergence Theorem (see, for example, Royden, p.89), we conclude that the limit of $\int g(\underline{x}|d_1,d_2)d\underline{x}$ is $\int g(\underline{x})d\underline{x}$.

We now make use of assumption (2) to prove the above claims. First, it is an immediate consequence of the assumption that for sufficiently large d_1 and d_2,

$$g(\underline{x}|d_1,d_2) \leq h(\underline{x}|d_1,d_2) = \prod_{k=1}^{2} \{1 + (\underline{x}-\underline{\mu}_k)^T\underline{B}_k(\underline{x}-\underline{\mu}_k)/d_k\}^{-d_k/2}.$$

Also, it can be directly verified that as $d_1, d_2 \to \infty$, $h(\underline{x}|d_1,d_2)$ monotonically decreases to an integrable function

$$h(\underline{x}) = \exp\{-(\underline{\mu}_1-\underline{\mu}_2)^T\underline{B}_1(\underline{B}_1+\underline{B}_2)^{-1}\underline{B}_2(\underline{\mu}_1-\underline{\mu}_2)/2\}$$

$$\times \exp\{-(\underline{x}-\underline{\nu})^T(\underline{B}_1+\underline{B}_2)(\underline{x}-\underline{\nu})/2\},$$

where $\underline{\nu} = (\underline{B}_1+\underline{B}_2)^{-1}(\underline{B}_1\underline{\mu}_1+\underline{B}_2\underline{\mu}_2)$. To establish the integrability of $h(\underline{x}|d_1d_2)$, let $m = \min(d_1,d_2)$. Then the monotonicity of the function implies that

$$h(\underline{x}d_1,d_2) \le \prod_{k=1}^{2}\{1+(\underline{x}-\underline{\mu}_k)^T\underline{B}_k(\underline{x}-\underline{\mu}_k)/m\}^{-m/2}$$

$$\le\{1+(\underline{x}-\underline{\nu})^T(\underline{B}_1+\underline{B}_2)(\underline{x}-\underline{\nu})/m\}^{-m/2}$$

For sufficiently large d_1 and d_2, the t-like factor on the right-hand side, denoted by $h_m(\underline{x})$, is proportional to a multivariate t density with m-p degrees of freedom, mean $\underline{\nu}$ and scale matrix $\underline{B}_1+\underline{B}_2$; therefore, $h(\underline{x}|d_1,d_2)$ is integrable. Let m be fixed. Define $\phi(\underline{x}|d_1,d_2) = h_m(\underline{x}) - h(\underline{x}|d_1,d_2)$, for $d_1, d_2 > m$. Applying the Monotone Convergence Theorem to this sequence of functions together with the fact that $h_m(\underline{x})$ is integrable, we conclude that $\int h(\underline{x}|d_1,d_2)d\underline{x}$ tends to $\int h(\underline{x})d\underline{x}$). This completes the proof.

We point out that if only the elements of a subset of $d_1,\ldots,d_L$ tend to infinity, then the theorem will apply to the product of the corresponding t-like factors while the remaining factors will stay put.

We now consider a special form of the poly-t density (1) with scale matrix

$$\underline{M}_k(d_k) = w_k(d_k)\underline{N}_k \tag{3}$$

for some positive function w_k and semi-positive definite matrix $\underline{N}_k$. The following corollary concerns the asymptotic behavior of such a poly-t density.

Corollary For the particular poly-t density with $\underline{M}_k(d_k)$ of the form (3), $k=1,\ldots,L$, assume that $d_k w_k(d_k) \to c_k > 0$ as $d_k \to \infty$, $k=1,\ldots,L$ and $\underline{M} = \underline{M}_1+\ldots+\underline{M}_L > 0$,

where $\underline{M}_k = c_k \underline{N}_k$ is the limit of $d_k \underline{M}_k(d_k)$. Then $f(\underline{x}|d_1,\ldots,d_L) \to f_N(\underline{x}|\underline{\mu},\underline{M}^{-1})$ as $d_1,\ldots,d_L \to \infty$.

Proof We need to check that the two asusmptions of the theorem hold for this particular form of poly-t density. Assumption (1) is obviously satisfied. For assumption (2), we take $\underline{B}_k = \underline{\varepsilon}_k \underline{M}_k$ for some appropiately chosen positive number ε_k. Clearly, $\underline{B}_1 + \ldots + \underline{B}_L = \varepsilon_1 \underline{M}_1 + \ldots + \varepsilon_L \underline{M}_L > 0$. We must choose ε_k such that $d_k \underline{M}_k(d_k) - \underline{B}_k = \{d_k w_k(d_k) - \varepsilon_k c_k\} \underline{N}_k > 0$ for d_k sufficiently large. Since $d_k w_k(d_k) \to c_k$ as $d_k \to \infty$, it follows that for d_k large enough, $d_k w_k(d_k) > c_k - n_k > 0$ where n_k is an arbitrary positive number. Therefore, we can choose $\varepsilon_k = (c_k - n_k)/c_k$. This completes the proof.

An example of a poly-t density with scale matrices of the specific form (3) is when d_k and $\underline{M}_k(d_k)$ are given by (2). Here we may choose $w_k(d_k) = 1/(d_k - m_k)$, and $\underline{N}_k = \underline{M}_k$. Note that the $\underline{M}_k$ matrix in (2) is the limit of $d_k \underline{M}_k(d_k)$ since $c_k = 1$.

3. APPLICATION TO BAYESIAN LINEAR MODELING

The results of the previous section are now applied to a multiple regression model with an exchangeable normal prior on the regression coefficients and independent inverted gamma priors on the variance components. The sampling model is represented by

$$\underline{y} = \underline{X}\,\underline{\beta} + \underline{\varepsilon},$$
$$\underline{\varepsilon} \sim N(\underline{0}, \sigma^2 \underline{I}_n),$$

where $\underline{y}$ is an nx1 vector of observations, $\underline{X}$ is an nxp fixed design matrix in correlation form, and $\underline{I}_n$ is the nxn identity matrix. The prior specification is given by

$$\underline{\beta} \sim N(\mu \underline{1}_p, \sigma_\beta^2 \underline{I}_p),$$
$$\nu\lambda/\sigma^2 \sim \chi^2_\nu, \ \nu_\beta\lambda_\beta/\sigma_\beta^2 \sim \chi^2_{\nu_\beta},$$

where $\underline{1}_p$ is a px1 vector of ones, and the λ's and ν's are hyperparameters that determine the prior means and variances of the variance components. For example $E(\sigma^2) = \nu\lambda/(\nu-2)$, and $V(\sigma^2) = 2\nu^2\lambda^2/(\nu-2)^2(\nu-4)$. Therefore, a large value of reflects a strong prior certainty the λ is a correct guess of σ^2. Additionally, we assume that prior knowledge about μ is vague.

In the special case of the one-way random effects model with a block diagonal matrix $\underline{X} = \text{diag}(\underline{1}_{n_1},\ldots,\underline{1}_{n_p})$, where n_i is the sample size of the ith group, Hill (1965, 1977, 1980) presented exact and approximate posterior inference of σ_β^2 and σ^2. His results can be used to obtain the posterior moments of the group means $\underline{\beta}$. Lindley and Smith (1972) estimated the parameters of the above general model using a computationally simple posterior joint modal approach. None of these authors, however, discussed the approximation of the posterior density of $\underline{\beta}$, which we denote by $f(\underline{\beta}|\underline{y})$. In the following, we show how the results of Section 2 may be used to approximate this density.

From Lindley and smith, the posterior density of $\underline{\beta}$ is

$$f(\underline{\beta}|\underline{y}) \propto (1+\underline{\beta}^T\underline{H}\underline{\beta}/\nu_\beta\lambda_\beta)^{-(\nu_\beta+p-1)/2} \times \{1+(\underline{\beta}-\hat{\underline{\beta}})^T\underline{X}^T\underline{X}(\underline{\beta}-\hat{\underline{\beta}})/(\nu\lambda+ns^2)\}^{-(\nu+n)/2},$$

where $\underline{H} = \underline{I}_p-(1/p)\underline{1}_p\underline{1}_p^T$, $\hat{\underline{\beta}}$ is a least squares estimate of $\underline{\beta}$ and s^2 is the residual mean square. The posterior density of $\underline{\beta}$ is a poly-t density of the special form (2) with

$$\nu_1 = \nu_\beta, \ \nu_2 = \nu + ns^2/\lambda,$$
$$m_1 = p-1, \ m_2 = n - ns^2/\lambda,$$
$$\underline{M}_1 = \underline{H}/\lambda_\beta, \ M_2 = \underline{X}^T\underline{X}/\lambda.$$

When there is prior near certainty concerning σ_β^2 and σ^2 so that both ν_β and ν are large, a simple application of the corollary shows that $f(\underline{\beta}|\underline{y})$ is approximately normal with mean $\underline{\mu}$ and covariance matrix $\underline{\Sigma}$ given by

$$\underline{\mu} = (\underline{H}/\lambda_\beta + \underline{X}^T\underline{X}/\lambda)^{-1}\,\underline{X}^T\underline{y},$$
$$\underline{\Sigma} = (\underline{H}/\lambda_\beta + \underline{X}^T\underline{X}/\lambda)^{-1}.$$

When there is a considerable amount of uncertainty concerning the prior value λ_β (or λ) so that ν_β (or ν) is not assigned a large value, the normal approximation may be improved upon by using a generalization of the Tiao-Zellner expansion. An alternative was proposed by Dickey (1967) using an expansion based on Appell's polynomials (1880) in the univariate case (p=1); however, the multivariate version has not been studied. The original Tiao-Zellner expansion is designed for a special case of a poly-t density with L=2 proper multivariate t factors with mean $\underline{\mu}_k$, scale matrix $\underline{M}_k$ and ν_k degrees of freedom, k=1,2. In the expansion, the poly-t density is approximated by a density in the form of $f_N(\underline{x}|\underline{\mu},\underline{M}^{-1})$, where $\underline{\mu} = \underline{M}^{-1}(\underline{M}_1\underline{\mu}_1+\underline{M}_2\underline{\mu}_2)$ and $\underline{M} = \underline{M}_1 + \underline{M}_2$, multiplied by a double infinite series in the inverse powers of ν_1 and ν_2. When most of the probability mass is concentrated in the intersection of two ellipsoids $\{\underline{x}|\ (\underline{x}-\underline{\mu}_k)^T\underline{M}_k(\underline{x}-\underline{\mu}_k) \le \nu_k\}$, k=1,2, the expansion will yield reasonable approximation with only a few terms in the series expansion.

The Tiao-Zellner expansion may be generalized to the case of a poly-t density of the form (2). Lindley (1971) mentioned the use of this expansion in the case of the one-way model. However, he did not specify the limiting normal density. Also, the degrees of freedom ν_1 and ν_2 were not correctly stated. We now sketch the essential steps in the generalization using $f(\underline{\beta}|\underline{y})$ in (4) as an example.

Let ν_k and m_k be as defined in (5), $Q_1 = \underline{\beta}^T\underline{H}\underline{\beta}/\lambda_\beta$, and $Q_2 = (\underline{\beta}-\hat{\underline{\beta}})^T\underline{X}^T\underline{X}(\underline{\beta}-\hat{\underline{\beta}})/\lambda$. Following Tiao and Zellner, we obtain for the first factor

$$(1+Q_1/\nu_1)^{-(\nu_1+m_1)/2} = \exp(-Q_1/2)\exp[Q_1/2-\{(\nu_1+m_1)/2\}\log(1+Q_1/\nu_1)].$$

This is of the same form as that obtained in (3.2) of Tiao and Zellner except that we replace p there by m_1. Therefore, the functional forms of the polynomials $p_i = p_i(Q_1)$ in the expansion

$$(1+Q_1/\nu_1)^{-(\nu_1+m_1)/2} = \exp(-Q_1/2)\sum_{j=0}^{\infty} p_i\,\nu_1^{-i}$$

are of the same forms as those given in Tiao and Zellner except that the constant p is replaced by m_1 (for instance, $p_1 = (Q_1 - m_1 Q_1)/4$). Similarly, for the second factor,

$$(1+Q_2/\nu_2)^{-(\nu_2+m_2)/2} = \exp(-Q_2/2)\sum_{j=0}^{\infty} q_j\,\nu_2^{-j}$$

where the constant p which appears in the polynomials $q_j = q_j(Q_2)$ in Tiao and Zellner is now replaced by m_2.

Following the rest of the derivation of Tiao and zellner, we obtain the *generalized expansion*

$$f(\underline{\beta}|\underline{y}) = f_N(\underline{\beta}|\underline{\mu},\underline{M}^{-1})\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d_{ij}\nu_1^{-i}\nu_2^{-j},$$

where the leading normal term is the one guaranteed by the corollary, and the d_{ij}'s are polynomials in p_i and q_j given in (3.9) of Tiao and Zellner. The coefficients which appear in the polynomial d_{ij}, namely $b_{rs} = E\{p_r(Q_1)q_s(Q_2)\}$ for $r \leq i$, $s \leq j$, also have to be modified accordingly. The modification essentially consists of the following steps: (1) Change the constant p in $p_r(Q_1)$ given in Tiao and Zellner to m_1; (2) Change the constant p in $q_s(Q_2)$ given in Tiao and Zellner to m_2, and (3) Use the bivariate moment-cumulant inversion formulas as given by Cook (1951) to express b_{rs} in terms of the mixed cumulants of Q_1 and Q_2. Expressions for these cumulants are derived in Tiao and Zellner. Following these four steps, the coefficients b_{rs} can be evaluated

in a straightforward manner. We omit the details here.

In the case of a product of two t-like factors of the form (2), it is also feasible to approximate the density by direct one-dimensional numerical integration. Dickey (1968) showed that integration can be facilitated if the scale matrices are simultaneously diagonalized first (more details are presented in Box and Tiao (1973, chapter 9)). The generalized Tiao-Zellner expansion offers a simple approximation alternative in terms of familiar multivariate normal calculations. When there are many such t-like factors, such as in the posterior analysis of a general mixed linear model with many variance components, it will be virtually impossible to approximate the density by high-dimensional numerical integration. The generalized expansion, however, can be extended to the case of many t-like factors in a straightforward manner. Obviously, the computational complexity will increase rapidly as the number of t-like factors increases.

ACKNOWLEDGEMENTS

Preparation of this article was supported by the office of Naval Research Contract No. N00014-85-K-0485.

BIBLIOGRAPHY

Appell, P. (1880). Dévelopment in série entière de $(1+ax)^{1/x}$. Archiv der Mathematik und Physik, 65, 171-175.

Box, G.E.P. and Tiao, G.C. (1973). Bayesian Inference in Statistical Analysis, Reading: Addison-Wesley.

Broemeling, L.D. (1985). Bayesian Analysis of Linear Models, New York: Marcel-Dekker.

Broemeling, L.D. and Abdullah, M.Y. (1984). An approximation to the poly-t distribution. Communciations in Statistics A,11, 1407-1422.

Cook, M.B. (1975). Bivariate κ-statistics and cumulants of their joint sampling distribution. Biometrika, 38, 179-195.

Dickey, J.M. (1967). Expansions of t densities and related complete integrals. Annals of Mathematical Statistics, 38, 503-510.

_____(1968). Three multidimensional-integral identities with Bayesian applications. Annals of Mathematical Statistics, 39, 1615-1627.

_____(1974). Bayesian alternatives to the F-test and least-squares estimate in the normal linear model. Studies in Bayesian Econometrics and Statistics eds. S.E. Fienberg and A. Zellner, New York: North Holland.

Drèze, J.H. (1977). Bayesian analysis using poly-t densities. New Developments in the Applications of Bayesian Methods, eds. A. Aykac and C. Brumat, New York: North Holland.

Hill, B.M. (1965). Inference about variance components in the one-way model. Journal of the American Statistical Association, 60, 806-825.

_____(1977). Exact and approximate Bayesian solutions for inference about variance components and multivariate inadmissibility. New Developments in the Applications of Bayesian Methods, eds. A. Aykac and C. Brumat, New York: North Holland.

_____(1980). Robust analysis of the random model and weighted least squares regression. Evaluation of Econometric Models, ed. J. Kementa, New York: Academic Press.

Lindley, D.V. (1971). The estimation of many parameters. Foundation of Statistical Inference, eds. V.P. Godambe and D. A. Sprott, Toronto: Holt, Reinhart and Winston.

Lindley, D.V. and Smith, A. F. M. (1972). Bayes estimates for the linear model. Journal of the Royal Statistical Society, Series B, 43, 1-41.

Rajagopalan, M. and Broemeling, L.D. (1983). Bayesian inference for the variance components in general mixed linear models. Communications in Statistics A,12, 701-724.

Royden, H.L. (1970). Real Analysis, New York: Macmillan.

Scheffé, H. (1947). A useful convergence theorem for probability distributions. Annals of Mathematical Statistics, 18, 434-458.

Tiao, G.C. and Zellner, A. (1964). Bayes's theorem and the use of prior knowledge in regression analysis. Biometrika, 51, 219-230.

Sloan-Kettering Institute/Wong

Dr. Terry Ackerman
American College Testing Programs
P.O. Box 168
Iowa City, IA 52243

Dr. Robert Ahlers
Code N711
Human Factors Laboratory
Naval Training Systems Center
Orlando, FL 32813

Dr. James Algina
University of Florida
Gainesville, FL 32605

Dr. Erling B. Andersen
Department of Statistics
Studiestraede 6
1455 Copenhagen
DENMARK

Dr. Eva L. Baker
UCLA Center for the Study
of Evaluation
145 Moore Hall
University of California
Los Angeles, CA 90024

Dr. Isaac Bejar
Educational Testing Service
Princeton, NJ 08450

Dr. Menucha Birenbaum
School of Education
Tel Aviv University
Tel Aviv, Ramat Aviv 69978
ISRAEL

Dr. Arthur S. Blaiwes
Code N711
Naval Training Systems Center
Orlando, FL 32813

Dr. Bruce Bloxom
Defense Manpower Data Center
550 Camino El Estero,
Suite 200
Monterey, CA 93943-3231

Dr. R. Darrell Bock
University of Chicago
NORC
6030 South Ellis
Chicago, IL 60637

Cdt. Arnold Bohrer
Sectie Psychologisch Onderzoek
Rekruterings-En Selectiecentrum
Kwartier Koningen Astrid
Bruijnstraat
1120 Brussels, BELGIUM

Dr. Robert Breaux
Code N-095R
Naval Training Systems Center
Orlando, FL 32813

Dr. Robert Brennan
American College Testing
Programs
P. O. Box 168
Iowa City, IA 52243

Dr. Lyle D. Broemeling
ONR Code 1111SP
800 North Quincy Street
Arlington, VA 22217

Mr. James W. Carey
Commandant (G-PTE)
U.S. Coast Guard
2100 Second Street, S.W.
Washington, DC 20593

Dr. James Carlson
American College Testing
Program
P.O. Box 168
Iowa City, IA 52243

Dr. John B. Carroll
409 Elliott Rd.
Chapel Hill, NC 27514

Dr. Robert Carroll
OP 0187
Washington, DC 20370

Mr. Raymond E. Christal
AFHRL/MOE
Brooks AFB, TX 78235

Dr. Norman Cliff
Department of Psychology
Univ. of So. California
University Park
Los Angeles, CA 90007

Director,
Manpower Support and
Readiness Program
Center for Naval Analysis
2000 North Beauregard Street
Alexandria, VA 22311

Dr. Stanley Collyer
Office of Naval Technology
Code 222
800 N. Quincy Street
Arlington, VA 22217-5000

Dr. Hans Crombag
University of Leyden
Education Research Center
Boerhaavelaan 2
2334 EN Leyden
The NETHERLANDS

Mr. Timothy Davey
University of Illinois
Educational Psychology
Urbana, IL 61801

Dr. Doug Davis
Chief of Naval Education
and Training
Naval Air Station
Pensacola, FL 32508

Dr. Dattprasad Divgi
Center for Naval Analysis
4401 Ford Avenue
P.O. Box 16268
Alexandria, VA 22302-0268

Dr. Hei-Ki Dong
Ball Foundation
800 Roosevelt Road
Building C, Suite 206
Glen Ellyn, IL 60137

Dr. Fritz Drasgow
University of Illinois
Department of Psychology
603 E. Daniel St.
Champaign, IL 61820

Defense Technical
Information Center
Cameron Station, Bldg 5
Alexandria, VA 22314
Attn: TC
(12 Copies)

Dr. Stephen Dunbar
Lindquist Center
for Measurement
University of Iowa
Iowa City, IA 52242

Dr. James A. Earles
Air Force Human Resources Lab
Brooks AFB, TX 78235

Dr. Kent Eaton
Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

Dr. John M. Eddins
University of Illinois
252 Engineering Research
Laboratory
103 South Mathews Street
Urbana, IL 61801

Dr. Susan Embretson
University of Kansas
Psychology Department
426 Fraser
Lawrence, KS 66045

Dr. Benjamin A. Fairbank
Performance Metrics, Inc.
5825 Callaghan
Suite 225
San Antonio, TX 78228

Dr. Pat Federico
Code 511
NPRDC
San Diego, CA 92152-6800

Sloan-Kettering Institute/Wong

Dr. Leonard Feldt
Lindquist Center
for Measurement
University of Iowa
Iowa City, IA 52242

Dr. Richard L. Ferguson
American College Testing
Program
P.O. Box 168
Iowa City, IA 52240

Dr. Gerhard Fischer
Liebiggasse 5/3
A 1010 Vienna
AUSTRIA

Dr. Myron Fischl
Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

Prof. Donald Fitzgerald
University of New England
Department of Psychology
Armidale, New South Wales 2351
AUSTRALIA

Mr. Paul Foley
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Alfred R. Fregly
AFOSR/NL
Bolling AFB, DC 20332

Dr. Robert D. Gibbons
University of Illinois-Chicago
P.O. Box 6998
Chicago, IL 69680

Dr. Janice Gifford
University of Massachusetts
School of Education
Amherst, MA 01003

Dr. Robert Glaser
Learning Research
& Development Center
University of Pittsburgh
3939 O'Hara Street
Pittsburgh, PA 15260

Dr. Bert Green
Johns Hopkins University
Department of Psychology
Charles & 34th Street
Baltimore, MD 21218

Dipl. Pad. Michael W. Habon
Universitat Dusseldorf
Erziehungswissenschaftliches
Universitatsstr. 1
D-4000 Dusseldorf 1
WEST GERMANY

Dr. Ronald K. Hambleton
Prof. of Education & Psychology
University of Massachusetts
at Amherst
Hills House
Amherst, MA 01003

Dr. Delwyn Harnisch
University of Illinois
51 Gerty Drive
Champaign, IL 61820

Ms. Rebecca Hetter
Navy Personnel R&D Center
Code 62
San Diego, CA 92152-6800

Dr. Paul W. Holland
Educational Testing Service
Rosedale Road
Princeton, NJ 08541

Prof. Lutz F. Hornke
Institut fur Psychologie
RWTH Aachen
Jaegerstrasse 17/19
D-5100 Aachen
WEST GERMANY

Dr. Paul Horst
677 G Street, #184
Chula Vista, CA 90010

Mr. Dick Hoshaw
OP-135
Arlington Annex
Room 2834
Washington, DC 20350

Dr. Lloyd Humphreys
University of Illinois
Department of Psychology
603 East Daniel Street
Champaign, IL 61820

Dr. Steven Hunka
Department of Education
University of Alberta
Edmonton, Alberta
CANADA

Dr. Huynh Huynh
College of Education
Univ. of South Carolina
Columbia, SC 29208

Dr. Robert Jannarone
Department of Psychology
University of South Carolina
Columbia, SC 29208

Dr. Dennis E. Jennings
Department of Statistics
University of Illinois
1409 West Green Street
Urbana, IL 61801

Dr. Douglas H. Jones
Thatcher Jones Associates
P.O. Box 6640
10 Trafalgar Court
Lawrenceville, NJ 08648

Dr. Milton S. Katz
Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

Prof. John A. Keats
Department of Psychology
University of Newcastle
N.S.W. 2308
AUSTRALIA

Dr. G. Gage Kingsbury
Portland Public Schools
Research and Evaluation Department
501 North Dixon Street
P. O. Box 3107
Portland, OR 97209-3107

Dr. William Koch
University of Texas-Austin
Measurement and Evaluation
Center
Austin, TX 78703

Dr. James Krantz
Computer-based Education
Research Laboratory
University of Illinois
Urbana, IL 61801

Dr. Leonard Kroeker
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Daryll Lang
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Thomas Leonard
University of Wisconsin
Department of Statistics
1210 West Dayton Street
Madison, WI 53705

Dr. Michael Levine
Educational Psychology
210 Education Bldg.
University of Illinois
Champaign, IL 61801

Dr. Charles Lewis
Educational Testing Service
Princeton, NJ 08541

Dr. Robert Linn
College of Education
University of Illinois
Urbana, IL 61801

Dr. Robert Lockman
Center for Naval Analysis
4401 Ford Avenue
P.O. Box 16268
Alexandria, VA 22302-0268

Dr. Frederic M. Lord
Educational Testing Service
Princeton, NJ 08541

Sloan-Kettering Institute/Wong

Dr. James Lumsden
Department of Psychology
University of Western Australia
Nedlands W.A. 6009
AUSTRALIA

Dr. Milton Maier
Center for Naval Analysis
4401 Ford Avenue
P.O. Box 16268
Alexandria, VA 22302-0268

Dr. William L. Maloy
Chief of Naval Education
and Training
Naval Air Station
Pensacola, FL 32508

Dr. Gary Marco
Stop 31-E
Educational Testing Service
Princeton, NJ 08451

Dr. Clessen Martin
Army Research Institute
5001 Eisenhower Blvd.
Alexandria, VA 22333

Dr. James McBride
Psychological Corporation
c/o Harcourt, Brace,
Javanovich Inc.
1250 West 6th Street
San Diego, CA 92101

Dr. Clarence McCormick
HQ, MEPCOM
MEPCT-P
2500 Green Bay Road
North Chicago, IL 60064

Dr. Robert McKinley
Educational Testing Service
Princeton, NJ 08541

Dr. James McMichael
Technical Director
Navy Personnel R&D Center
San Diego, CA 92152

Dr. Barbara Means
Human Resources
Research Organization
1100 South Washington
Alexandria, VA 22314

Dr. Robert Mislevy
Educational Testing Service
Princeton, NJ 08541

Dr. William Montague
NPRDC Code 13
San Diego, CA 92152-6800

Ms. Kathleen Moreno
Navy Personnel R&D Center
Code 62
San Diego, CA 92152-6800

Headquarters, Marine Corps
Code MPI-20
Washington, DC 20380

Dr. W. Alan Nicewander
University of Oklahoma
Department of Psychology
Oklahoma City, OK 73069

Deputy Technical Director
NPRDC Code 01A
San Diego, CA 92152-6800

Director, Training Laboratory,
NPRDC (Code 05)
San Diego, CA 92152-6800

Director, Manpower and Personnel
Laboratory,
NPRDC (Code 06)
San Diego, CA 92152-6800

Director, Human Factors
& Organizational Systems Lab,
NPRDC (Code 07)
San Diego, CA 92152-6800

Fleet Support Office,
NPRDC (Code 301)
San Diego, CA 92152-6800

Library, NPRDC
Code P201L
San Diego, CA 92152-6800

Sloan-Kettering Institute/Wong

Commanding Officer,
Naval Research Laboratory
Code 2627
Washington, DC 20390

Dr. Harold F. O'Neil, Jr.
School of Education - WPH 801
Department of Educational
Psychology & Technology
University of Southern California
Los Angeles, CA 90089-0031

Dr. James Olson
WICAT, Inc.
1875 South State Street
Orem, UT 84057

Office of Naval Research,
Code 1142PT
800 N. Quincy Street
Arlington, VA 22217-5000
(6 Copies)

Office of Naval Research,
Code 125
800 N. Quincy Street
Arlington, VA 22217-5000

Assistant for MPT Research,
Development and Studies
OP 01B7
Washington, DC 20370

Dr. Judith Orasanu
Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

Dr. Jesse Orlansky
Institute for Defense Analyses
1801 N. Beauregard St.
Alexandria, VA 22311

Dr. Randolph Park
Army Research Institute
5001 Eisenhower Blvd.
Alexandria, VA 22333

Wayne M. Patience
American Council on Education
GED Testing Service, Suite 20
One Dupont Circle, NW
Washington, DC 20036

Dr. James Paulson
Department of Psychology
Portland State University
P.O. Box 751
Portland, OR 97207

Administrative Sciences Department,
Naval Postgraduate School
Monterey, CA 93940

Department of Operations Research,
Naval Postgraduate School
Monterey, CA 93940

Dr. Mark D. Reckase
ACT
P. O. Box 168
Iowa City, IA 52243

Dr. Malcolm Ree
AFHRL/MP
Brooks AFB, TX 78235

Dr. Barry Riegelhaupt
HumRRO
1100 South Washington Street
Alexandria, VA 22314

Dr. Carl Ross
CNET-PDCD
Building 90
Great Lakes NTC, IL 60088

Dr. J. Ryan
Department of Education
University of South Carolina
Columbia, SC 29208

Dr. Fumiko Samejima
Department of Psychology
University of Tennessee
Knoxville, TN 37916

Mr. Drew Sands
NPRDC Code 62
San Diego, CA 92152-6800

Lowell Schoer
Psychological & Quantitative
Foundations
College of Education
University of Iowa
Iowa City, IA 52242

Dr. Mary Schratz
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Dan Segall
Navy Personnel R&D Center
San Diego, CA 92152

Dr. W. Steve Sellman
OASD(MRA&L)
2B269 The Pentagon
Washington, DC 20301

Dr. Kazuo Shigemasu
7-9-24 Kugenuma-Kaigan
Fujusawa 251
JAPAN

Dr. William Sims
Center for Naval Analysis
4401 Ford Avenue
P.O. Box 16268
Alexandria, VA 22302-0268

Dr. H. *Wallace Sinaiko*
Manpower Research
and Advisory Services
Smithsonian Institution
801 North Pitt Street
Alexandria, VA 22314

Dr. Richard E. Snow
Department of Psychology
Stanford University
Stanford, CA 94306

Dr. Richard Sorensen
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Paul Speckman
University of Missouri
Department of Statistics
Columbia, MO 65201

Dr. Judy Spray
ACT
P.O. Box 168
Iowa City, IA 52243

Dr. Martha Stocking
Educational Testing Service
Princeton, NJ 08541

Dr. Peter Stoloff
Center for Naval Analysis
200 North Beauregard Street
Alexandria, VA 22311

Dr. William Stout
University of Illinois
Department of Mathematics
Urbana, IL 61801

Maj. Bill Strickland
AF/MPXOA
4E168 Pentagon
Washington, DC 20330

Dr. Hariharan Swaminathan
Laboratory of Psychometric and
Evaluation Research
School of Education
University of Massachusetts
Amherst, MA 01003

Mr. Brad Sympson
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. John Tangney
AFOSR/NL
Bolling AFB, DC 20332

Dr. Kikumi Tatsuoka
CERL
252 Engineering Research
Laboratory
Urbana, IL 61801

Dr. Maurice Tatsuoka
220 Education Bldg
1310 S. Sixth St.
Champaign, IL 61820

Dr. David Thissen
Department of Psychology
University of Kansas
Lawrence, KS 66044

Mr. Gary Thomasson
University of Illinois
Educational Psychology
Champaign, IL 61820

Dr. Robert Tsutakawa
University of Missouri
Department of Statistics
222 Math. Sciences Bldg.
Columbia, MO 65211

Dr. Ledyard Tucker
University of Illinois
Department of Psychology
603 E. Daniel Street
Champaign, IL 61820

Dr. Vern W. Urry
Personnel R&D Center
Office of Personnel Management
1900 E. Street, NW
Washington, DC 20415

Dr. David Vale
Assessment Systems Corp.
2233 University Avenue
Suite 310
St. Paul, MN 55114

Dr. Frank Vicino
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Howard Wainer
Division of Psychological Studies
Educational Testing Service
Princeton, NJ 08541

Dr. Ming-Mei Wang
Lindquist Center
for Measurement
University of Iowa
Iowa City, IA 52242

Dr. Thomas A. Warm
Coast Guard Institute
P. O. Substation 18
Oklahoma City, OK 73169

Dr. Brian Waters
Program Manager
Manpower Analysis Program
HumRRO
1100 S. Washington St.
Alexandria, VA 22314

Dr. David J. Weiss
N660 Elliott Hall
University of Minnesota
75 E. River Road
Minneapolis, MN 55455

Dr. Ronald A. Weitzman
NPS, Code 54Wz
Monterey, CA 92152-6800

Major John Welsh
AFHRL/MOAN
Brooks AFB, TX 78223

Dr. Douglas Wetzel
Code 12
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Rand R. Wilcox
University of Southern
California
Department of Psychology
Los Angeles, CA 90007

German Military Representative
ATTN: Wolfgang Wildegrube
Streitkraefteamt
D-5300 Bonn 2
4000 Brandywine Street, NW
Washington, DC 20016

Dr. Bruce Williams
Department of Educational
Psychology
University of Illinois
Urbana, IL 61801

Dr. Hilda Wing
Psychological Corporation
c/o Harcourt, Brace,
Javanovich Inc.
1250 West 6th Street
San Diego, CA 92101

Dr. Martin F. Wiskoff
Navy Personnel R & D Center
San Diego, CA 92152-6800

Mr. John H. Wolfe
Navy Personnel R&D Center
San Diego, CA 92152-6800

Sloan-Kettering Institute/Wong

Dr. George Wong
Biostatistics Laboratory
Memorial Sloan-Kettering
Cancer Center
1275 York Avenue
New York, NY 10021

Dr. Wallace Wulfeck, III
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Kentaro Yamamoto
Computer-based Education
Research Laboratory
University of Illinois
Urbana, IL 61801

Dr. Wendy Yen
CTB/McGraw Hill
Del Monte Research Park
Monterey, CA 93940

Dr. Joseph L. Young
Memory & Cognitive
Processes
National Science Foundation
Washington, DC 20550

END

12-87

DTIC